

MULTIPLIER INFINITESIMAL BIALGEBRAS AND DERIVATOR LIE BIALGEBRAS

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ABSTRACT. We propose a definition of *multiplier infinitesimal bialgebra* and a definition of *derivator Lie bialgebra*. We give some examples of these structures and prove that every *bibalanced* multiplier infinitesimal bialgebra gives rise to a multiplier Lie bialgebra.

INTRODUCTION

An infinitesimal bialgebra is a triple (A, m, Δ) where (A, m) is an associative algebra, (A, Δ) is a coassociative coalgebra and for each $a, b \in A$,

$$\Delta(ab) = \sum ab_1 \otimes b_2 + a_1 \otimes a_2b.$$

Infinitesimal bialgebras were introduced by Joni and Rota [JR] in order to provide an algebraic framework for the calculus of divided differences. In [A1] the author introduces several new examples, in particular he shows that the path algebra of an arbitrary quiver admits a canonical structure of infinitesimal bialgebra.

The notion of *multiplier Hopf algebra* was introduced by Van Daele [VD1]. This concept is a far generalization of the usual concept of Hopf algebra to the non-unital case. In that work the author consider an associative algebra A , with or without identity, and a morphism Δ from A to the multiplier algebra $M(A \otimes A)$ of $A \otimes A$, and imposes certain conditions on Δ (such as *coassociativity*). The motivating example was the case where A is the algebra of finitely supported, complex valued functions on an infinite discrete group, and where $\Delta(f, g)(s, t) = f(st)$. In these and subsequent works Van Daele together with his collaborators have developed many of the results about finite dimensional Hopf algebras to this context.

In this work we introduce the notion of *multiplier infinitesimal bialgebra* (see definition 1.2) and give several examples of the concept. This is a mix of the two mentioned structures in which the coproduct take values in the multiplier algebra $M(A \otimes A)$ and at the same time it is a derivation, in which $M(A \otimes A)$ is consider as an A -bimodule in the obvious way.

We also introduce the notion of *derivator Lie bialgebra* and give a couple of examples of this structure. This is an approach to a nonassociative version of the multiplier Hopf algebras of Van Daele. In the last section we obtain a result that relates the two above concepts in a similar way to the one obtained by Aguiar in [A2].

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1. INFINITESIMAL MULTIPLIER BIALGEBRAS

From now on we only consider \mathbb{C} -algebras with non-degenerate product, that is, $\cdot : A \times A \rightarrow A$, the product of A , is no degenerate as an A -valued \mathbb{C} -bilinear form.

If A is an algebra, we will denote by $L(A, \cdot)$ (resp. $R(A, \cdot)$) the space of *left multipliers* (resp. *right multipliers*) of A :

$$\begin{aligned} L(A, \cdot) &= \{\lambda \in \text{Hom}_{\mathbb{C}}(A, A) : \lambda(ab) = \lambda(a)b, \forall a, b \in A\}, \\ R(A, \cdot) &= \{\rho \in \text{Hom}_{\mathbb{C}}(A, A) : \rho(ab) = a\rho(b), \forall a, b \in A\}. \end{aligned}$$

we also denote by $M(A, \cdot)$ the algebra of multipliers of A , i.e.

$$M(A, \cdot) = \{(\lambda, \rho) \in \text{Hom}_{\mathbb{C}}(A, A)^2 : \lambda \in L(A), \rho \in R(A) \text{ and } b\lambda(c) = \rho(b)c, \forall b, c \in A\}.$$

Remark 1.1. Let $a \in A$, define $\lambda_a, \rho_a \in \text{Hom}_{\mathbb{C}}(A, A)$ by $\lambda_a(b) = ab$ and $\rho_a(b) = ba$. Then λ_a and ρ_a are left and right multipliers, respectively. Moreover, since the product of A is non-degenerated, the canonical map $a \mapsto (\lambda_a, \rho_a)$ is a one to one algebra map from A to $M(A, \cdot)$.

We remind that the product in $M(A)$ is given by:

$$(\lambda, \rho)(\lambda', \rho') = (\lambda \circ \lambda', \rho' \circ \rho).$$

With this product, $M(A)$ is an algebra with identity.

Definition 1.1. A linear map $\Delta : A \rightarrow M(A \otimes A)$ is called a *coproduct* if

- (a) $T_3(a \otimes b) = \Delta(b)(a \otimes 1)$ and $T_4(a \otimes b) = (1 \otimes b)\Delta(a)$ are elements of $A \otimes A$ for all $a, b \in A$,
- (b) the map Δ is coassociative in the sense that:

$$(1) \quad (\mathbf{1} \otimes T_4) \circ (T_3 \otimes \mathbf{1}) = (T_3 \otimes \mathbf{1}) \circ (\mathbf{1} \otimes T_4).$$

Remark 1.2. Using the Sweedler notation, introduced by Van Daele, the identity (1) could be read as:

$$(2) \quad b_1 a \otimes b_{21} \otimes c b_{22} = b_{11} a \otimes b_{12} \otimes c b_2,$$

since

$$\begin{aligned}
(\mathbf{1} \otimes T_4) \circ (T_3 \otimes \mathbf{1})(a \otimes b \otimes c) &= (\mathbf{1} \otimes T_4) \circ (\Delta(b)(a \otimes 1) \otimes c) \\
&= (\mathbf{1} \otimes T_4) \circ (b_1 a \otimes b_2 \otimes c) \\
&= b_1 a \otimes (1 \otimes c) \Delta(b_2) \\
&= b_1 a \otimes b_{21} \otimes c b_{22}. \\
(T_3 \otimes \mathbf{1}) \circ (\mathbf{1} \otimes T_4)(a \otimes b \otimes c) &= (T_3 \otimes \mathbf{1}) \circ (a \otimes (1 \otimes c) \Delta(b)) \\
&= (T_3 \otimes \mathbf{1}) \circ (a \otimes b_1 \otimes c b_3) \\
&= \Delta(b_1)(a \otimes 1) \otimes c b_2 \\
&= b_{11} a \otimes b_{12} \otimes c b_2.
\end{aligned}$$

The common value in (2) will be denoted by $b_1 a \otimes b_2 \otimes c b_3$.

Definition 1.2. An *infinitesimal multiplier bialgebra* (or multiplier ε -bialgebra) is a triple (A, m, Δ) where

- (1) (A, m) is an associative algebra with non-degenerate product,
- (2) $\Delta : A \rightarrow M(A \otimes A)$ is a coproduct on A , and
- (3) $\Delta(ab) = \Delta(a) \circ (1 \otimes b) + (a \otimes 1) \circ \Delta(b)$.

1.1. Examples of Multipliers ε -Bialgebras.

Example 1.1. Quivers. Let (Q, s, t) be an infinite quiver. Where s and t stands for the source and target maps respectively. Let $A(Q)$ and $V(Q)$ be the set of arrows and vertices of Q respectively. Let

$$\Gamma = \{\gamma : I \rightarrow A(Q) : \text{where } I \subseteq \mathbb{Z}, \text{ is an interval and } t(\gamma(i)) = s(\gamma(i+1))\},$$

be the set of paths in Q (note that here we are considering finite and infinite paths).

For all $\gamma \in \Gamma$ we will denote $I_\gamma = \gamma^{-1}(A(Q))$. If $I_\gamma = [i, \cdot)$ then we write $s(\gamma) = \gamma_i = \gamma(i)$, the *source* of γ ; and if $I_\gamma = (-\infty, \cdot)$ we simply write $s(\gamma) = -\infty$. In an analogous way, if $\gamma \in \Gamma$ and $I_\gamma = (\cdot, i]$, we denote $t(\gamma) = \gamma_i = \gamma(i)$, the *target* of γ . and if $I_\gamma = (\cdot, +\infty)$ we write $t(\gamma) = +\infty$. The length of a path γ is defined as $|\gamma| := |I_\gamma|$, the cardinal of the set I_γ .

For any vertex $e \in V(Q)$ we will define two sets

$$\Gamma_e = \{i \in \text{Dom} \gamma : s(\gamma_i) = e\} \text{ and } \Gamma^e = \{i \in \text{Dom} \gamma : t(\gamma_i) = e\}.$$

Let $\Gamma' = \{\gamma \in \Gamma : \forall e \in V(Q), |\Gamma_e| < \infty \text{ and } |\Gamma^e| < \infty\} \cup \{\pm\infty\}$ for all $e \in V(Q)\} \cup \{-\infty, +\infty\}$. Thus Γ' is the set of paths $\gamma \in \Gamma$, such that for every $e \in V(Q)$, γ pass trough e only finitely times and we adjoin to this set two elements $\{-\infty, +\infty\}$ that doesn't belong to $V(Q)$ and we define $s(\pm\infty) = t(\pm\infty) = \pm\infty$.

Definition 1.3. The *generalized path algebra* of Q is the vector space $k_\infty Q$ with basis Γ' and where the multiplication is the concatenation of paths whenever is possible and zero otherwise. The added symbols $\pm\infty$ are idempotents by definition.

Remark 1.3. We can observe here that when we define the set Γ' we adjoin two elements $\pm\infty$, the reason for this is that we want to have a no degenerated product in the generalized path algebra $k_\infty Q$.

We are going to define a coproduct on $k_\infty Q$ in the following way. If $\gamma = \cdots \gamma_i \gamma_{i+1} \cdots$, then

$$\Delta(\gamma) = \sum_{i \in \mathbb{Z}} \cdots \gamma_{i-2} \gamma_{i-1} \otimes \gamma_{i+1} \gamma_{i+2} \cdots;$$

that is, for $\gamma \in \Gamma$ we define $\Delta(\gamma) = (\Delta_\lambda(\gamma), \Delta_\rho(\gamma)) \in M(A \otimes A)$ by the formulas:

$$\Delta_\lambda(\gamma)(\gamma' \otimes \gamma'') = \sum_{i \in \mathbb{Z}} \cdots \gamma_{i-2} \gamma_{i-1} \gamma' \otimes \gamma_{i+1} \gamma_{i+2} \cdots \gamma'' .(*)$$

$$\Delta_\rho(\gamma)(\gamma' \otimes \gamma'') = \sum_{i \in \mathbb{Z}} \gamma' \cdots \gamma_{i-2} \gamma_{i-1} \otimes \gamma'' \gamma_{i+1} \gamma_{i+2} \cdots .(**)$$

Since Γ^e and Γ_e are finite (by definition of Γ') then both sums are well defined. We define $\Delta(\pm\infty) = 0$ and extend Δ linearly to the whole A . It is clear that Δ_λ and Δ_ρ are left and right multipliers, respectively, and that $(\gamma_1 \otimes \gamma_2)\Delta_\lambda(\gamma)(\gamma_3 \otimes \gamma_4) = \Delta_\rho(\gamma)(\gamma_1 \otimes \gamma_2)(\gamma_3 \otimes \gamma_4)$; thus $\Delta(\gamma) \in M(A \otimes A)$.

That Δ is a derivation, i.e.,

$$(3) \quad \Delta(\gamma \cdot \gamma') = (\gamma \otimes 1)\Delta(\gamma') + \Delta(\gamma)(1 \otimes \gamma'),$$

follows by direct verification from the possible form of the paths. For instance, if $\gamma = \cdots \gamma_{n-1} \gamma_n$ then, we have two cases.

- (1) If e_n the target of γ_n is different of e'_0 , the source of γ'_0 , both sides of (3) are zero.
- (2) If $e_n = e'_0$, then

$$\Delta(\gamma \cdot \gamma') = \sum_{i \leq n} (\cdots \gamma_{i-1} \otimes \gamma_{i+1} \cdots \gamma_n \gamma'_1 \gamma'_2 \cdots) + \sum_{1 \leq i} (\cdots \gamma_n \gamma'_1 \gamma'_2 \cdots \gamma'_{i-2} \gamma'_{i-1} \otimes \gamma'_{i+1} \gamma'_{i+2} \cdots).$$

On the other hand

$$\begin{aligned} (\gamma \otimes 1)\Delta(\gamma') &= (\gamma \otimes 1) \sum_{1 \leq i} (\gamma'_0 \cdots \gamma'_{i-1} \otimes \gamma'_{i+1} \gamma'_{i+2} \cdots) \\ &= \sum_{1 \leq i} (\cdots \gamma_{n-1} \gamma_n \gamma'_1 \cdots \gamma'_{i-1} \otimes \gamma'_{i+1} \gamma'_{i+2} \cdots), \end{aligned}$$

and

$$\begin{aligned} \Delta(\gamma)(1 \otimes \gamma') &= \sum_{i \leq n} (\cdots \gamma_{i-1} \otimes \gamma_{i+1} \cdots \gamma_n)(1 \otimes \gamma') \\ &= \sum_{i \leq n} (\cdots \gamma_{i-1} \otimes \gamma_{i+1} \cdots \gamma_n \gamma'_1 \cdots), \end{aligned}$$

so the desired conclusion follows.

It is not hard to check that $\Delta(\gamma)(\gamma' \otimes 1) \in A \otimes A$ and $(1 \otimes \gamma)\Delta(\gamma') \in A \otimes A$.

Example 1.2. Posets.

Let L be a poset and let \mathcal{P} be the set of subposets of L with maximum and minimum. If $P \in \mathcal{P}$ we denote by 1_P and 0_P the maximum and the minimum of P respectively.

Denote by $A_{\mathcal{P}}$ the k -vector space with base \mathcal{P} equipped with the following product:

$$P * Q = \begin{cases} P \cup Q & \text{if } 1_P = 0_Q \text{ with the inherited order,} \\ 0 & \text{otherwise.} \end{cases}$$

for every $P, Q \in \mathcal{P}$ and extend this product linearly to the whole $A_{\mathcal{P}}$.

Remark 1.4. The algebra $A_{\mathcal{P}}$ is nonunital but have local units. In fact, if $P_1, P_2, \dots, P_n \in \mathcal{P}$ then write $S = \{0_{P_1}, \dots, 0_{P_n}\}$ and $T = \{1_{P_1}, \dots, 1_{P_n}\}$. It is clear that $\sum_{u \in S} \{u\}$ and $\sum_{v \in T} \{v\}$ are left and right local units, respectively, for any linear combination of $\{P_1, P_2, \dots, P_n\}$. Thus we have that the product of $A_{\mathcal{P}}$ is no degenerated and the canonical map from $A_{\mathcal{P}}$ (or $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$) to its multipliers is injective. We also have that $(A_{\mathcal{P}})^2 = A_{\mathcal{P}}$.

Notation: if $P \in \mathcal{P}$ we will denote $P_0 = P - \{1_P\}$. For every $P \in \mathcal{P}$ and $x \in P_0$ we will to write $(-\infty, x]_P := \{y \in P : y \leq x\}$ and $[x, +\infty)_P := \{y \in P : x \leq y\}$. If $x \notin P_0$ then $(-\infty, x]_P := 0$ and $[x, +\infty)_P := 0$.

Let $\Delta : A_{\mathcal{P}} \rightarrow M(A_{\mathcal{P}} \otimes A_{\mathcal{P}})$ be the map given by

$$\Delta_{\lambda}(P)(Q \otimes R) = \sum_{x \in P_0} (-\infty, x] * Q \otimes [x, +\infty) * R$$

and

$$\Delta_{\rho}(P)(Q \otimes R) = \sum_{x \in P_0} Q * (-\infty, x] \otimes R * [x, +\infty)$$

It is clear that $\Delta_{\lambda}(P)$ and $\Delta_{\rho}(P)$ are left and right multipliers of $A_{\mathcal{P}} \otimes A_{\mathcal{P}}$ respectively, and it easy to check that $(P \otimes Q)\Delta_{\lambda}(T)(R \otimes S) = \Delta_{\rho}(T)(P \otimes Q)(R \otimes S)$.

Lemma 1.1. *Let $P, S \in \mathcal{P}$. Then,*

$$\Delta(P)(S \otimes 1) \in A_{\mathcal{P}} \otimes A_{\mathcal{P}} \quad \text{and} \quad (1 \otimes S)\Delta(P) \in A_{\mathcal{P}} \otimes A_{\mathcal{P}}.$$

Proof. We are going to show that if $P, S \in \mathcal{P}$ then

$$(\Delta_{\lambda}(P) \circ (S \otimes 1), (S \otimes 1) \circ \Delta_{\rho}(P)) = (\lambda_U, \rho_U),$$

where $U = (-\infty, 0_S]_P * S \otimes [0_S, +\infty)_P$. In fact, if $0_S \in P_0$ then

$$\begin{aligned} \Delta_{\lambda}(P) \circ (S \otimes 1)(X \otimes Y) &= \sum_{x \in P_0} (-\infty, x] * S * X \otimes [x, +\infty) * Y \\ &= (-\infty, 0_S] * S * X \otimes [0_S, +\infty) * Y, \quad (*) \\ (S \otimes 1) \circ \Delta_{\rho}(P)(X \otimes Y) &= \sum_{x \in P_0} X * (-\infty, x] * S \otimes Y * [x, +\infty) \\ &= X * (-\infty, 0_S] * S \otimes Y * [0_S, +\infty), \quad (**) \end{aligned}$$

and

$$\begin{aligned}\lambda_U(X \otimes Y) &= (-\infty, 0_S] * S * X \otimes [0_S, +\infty) * Y, \\ \rho_U(X \otimes Y) &= X * (-\infty, 0_S] * S \otimes Y * [0_S, +\infty).\end{aligned}$$

The left and right multiplier of U are identical to $(*)$ and $(**)$ respectively. In the same way we show that $(1 \otimes S)\Delta(P) \in A_{\mathcal{P}} \otimes A_{\mathcal{P}}$. \square

Now we will to probe that Δ is a derivation. If $P * Q = 0$, then $1_P \neq 0_Q$ and then follows that $0 = \Delta(P)(1 \otimes Q) = (P \otimes 1)\Delta(Q)$. If $P * Q \neq 0$ then $1_P = 0_Q$ and $(P * Q)_0 = P_0 \cup Q_0$, thus

$$\begin{aligned}\Delta(P * Q) &= \sum_{x \in (P * Q)_0} (-\infty, x]_{P * Q} \otimes [x, +\infty)_{P * Q} \\ &= \sum_{x \in P_0} (-\infty, x]_{P * Q} \otimes [x, +\infty)_{P * Q} + \sum_{x \in Q_0} (-\infty, x]_{P * Q} \otimes [x, +\infty)_{P * Q} \\ &= \sum_{x \in P_0} (-\infty, x]_P \otimes [x, +\infty)_{P * Q} + \sum_{x \in Q_0} (-\infty, x]_{P * Q} \otimes [x, +\infty)_Q \\ &= \sum_{x \in P_0} (-\infty, x]_P \otimes [x, +\infty)_P * Q + \sum_{x \in Q_0} P * (-\infty, x]_Q \otimes [x, +\infty)_Q \\ &= \Delta(P)(1 \otimes Q) + (P \otimes 1)\Delta(Q).\end{aligned}$$

2. DERIVATOR LIE BIALGEBRAS

Definition 2.1. A *derivator Lie bialgebra* is a collection $(\mathfrak{g}, [\cdot, \cdot], \delta, \zeta, T_1, T_2)$ where:

- $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra,
- $\delta, \zeta : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ are derivations and
- $T_1, T_2 : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ are linear maps.

These data subject to the following axioms:

- **Antisymmetry.** The linear maps δ, ζ are *antisymmetric*, that is, for all $x \in \mathfrak{g}$

$$\tau \circ \delta_a = -\delta_a \text{ and } \tau \circ \zeta^b = -\zeta^b.$$

- **Generalized CoJacobi.** For all $a, b, x \in \mathfrak{g}$,

$$(4) \quad (Id + \sigma + \sigma^2)(\zeta^b \otimes Id)(T_1(a \otimes x)) = (Id + \sigma + \sigma^2)(\delta_a \otimes Id)(\tau(T_2(x \otimes b))).$$

Where ζ^a and δ_b stand for $\zeta(a)$ and $\delta(b)$ respectively, $\sigma = (Id \otimes \tau)(\tau \otimes Id)$ is the cyclic permutation and $\tau : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the flip map.

The pair (δ, ζ) will be called the *co-bracket* of \mathfrak{g} and the maps T_1, T_2 *intertwining operators*.

2.1. First examples of derivator Lie bialgebras. We remind here the definition of Lie bialgebra.

Definition 2.2. A *Lie bialgebra* over k , is a collection $(\mathfrak{g}, [\cdot, \cdot], \delta)$ where $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie k -algebra, $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is a derivation and it satisfies the following conditions,

- **(Antisymmetry)** $\tau \circ \delta = -\delta$ and
- **(CoJacobi)** $(Id + \sigma + \sigma^2)(\delta \otimes Id)\delta = 0$.

Example 2.1. If \mathfrak{g} is a one-dimensional Lie algebra all the Lie bialgebra structures are trivial, that is, all are equal to zero.

Example 2.2. Let \mathfrak{g} be a simple Lie algebra and Δ a root system for \mathfrak{g} . We define $r = \sum_{\alpha \in \Delta^+} e_\alpha \wedge e_{-\alpha}$. The internal derivation determined by r is a Lie bialgebra structure over \mathfrak{g} .

Example 2.3. Let $(\mathfrak{g}, [\cdot, \cdot], \beta)$ be a Lie bialgebra and consider $\alpha : \mathfrak{g} \rightarrow k$ a linear functional. We define

$$\delta, \zeta : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) \text{ by } \delta(x) = \zeta(x) = \alpha(x)\beta$$

and

$$T_1, T_2 : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \text{ by } T_1(a \otimes x) = \alpha(a)\beta(x) \text{ and } T_2(x \otimes b) = \alpha(b)\beta(x).$$

The collection $(\mathfrak{g}, [\cdot, \cdot], \delta, \zeta, T_1, T_2)$ is a derivator Lie bialgebra.

Example 2.4. Let \mathfrak{g} be a Lie algebra of dimension 2 and let $\{X, Y\}$ be a base of \mathfrak{g} such that $[X, Y] = X$. We consider the derivation $\beta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ defined by $\beta(X) = X \wedge Y$ and $\beta(Y) = 0$. Define a pair of operators $\iota_X : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ and $\iota_Y : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ by $\iota_X(Z) = Z \wedge X$ and $\iota_Y(Z) = Z \wedge Y$, respectively. We define $\delta = \zeta$ as $\delta(X) = \iota_Y$ and $\delta(Y) = X$, then we extend by linearity. Clearly δ is a $\text{Der}(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ valued derivation of \mathfrak{g} . If we take $T_1 = T_2 = \text{Id} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, then $(\mathfrak{g}, [\cdot, \cdot], \delta, \delta, \text{Id}, \text{Id})$ is a derivator Lie bialgebra; the verification is left to the reader. The only that we need to check is that each sides of equality (4) are equals to zero.

3. FROM INFINITESIMAL MULTIPLIER BIALGEBRAS TO DERIVATOR LIE BIALGEBRAS

In this section we describe a way to obtain derivator Lie bialgebras in a similar way in which in [A2] were obtained Lie bialgebras from infinitesimal bialgebras.

Definition 3.1. Let (A, μ, Δ) be an infinitesimal multiplier bialgebra. The *bibalanceator* of A is the linear map $\mathbb{B} : A \rightarrow \text{End}(A \otimes A)^2$, where $\mathbb{B}(a) = (\underline{\mathbb{B}}(a), \overline{\mathbb{B}}(a))$ with

$$(5) \quad \underline{\mathbb{B}}(a)(x \otimes y) = (x \otimes 1)\tau\Delta(y)(1 \otimes a) - \tau\Delta(y)(1 \otimes ax) + (1 \otimes y)\Delta(x)(a \otimes 1) - \Delta(x)(ay \otimes 1),$$

and

$$(6) \quad \overline{\mathbb{B}}(a)(x \otimes y) = (xa \otimes 1)\tau\Delta(y) - (a \otimes 1)\tau\Delta(y)(1 \otimes x) + (1 \otimes ya)\Delta(y) - (1 \otimes a)\Delta(x)(y \otimes 1).$$

We say that \mathbb{B} is *symmetric* if $\underline{\mathbb{B}}(a) = \underline{\mathbb{B}}(a) \circ \tau$ and $\overline{\mathbb{B}}(a) = \overline{\mathbb{B}}(a) \circ \tau$, for all $a \in A$. We say that A is *bibalanced* if its bibalanceator is equal to zero.

Proposition 3.1. Let (A, μ, Δ) be an infinitesimal multiplier bialgebra. We define the following linear maps

$$(7) \quad \delta : A \rightarrow \text{Hom}_k(A, A \otimes A), \quad x \mapsto \delta_a(x) := \Delta(x)(a \otimes 1) - \tau(\Delta(x)(a \otimes 1)),$$

$$(8) \quad \zeta : A \rightarrow \text{Hom}_k(A, A \otimes A), \quad x \mapsto \zeta^a(x) := (1 \otimes a)\Delta(x) - \tau((1 \otimes a)\Delta(x)).$$

Then,

$$(9) \quad \delta_a[x, y] = x \cdot \delta_a(y) - y \cdot \delta_a(x) + \underline{\mathbb{B}}(a)(y \otimes x) - \underline{\mathbb{B}}(a)(x \otimes y),$$

and

$$(10) \quad \zeta^a[x, y] = x \cdot \zeta^a(y) - y \cdot \zeta^a(x) + \overline{\mathbb{B}}(a)(y \otimes x) - \overline{\mathbb{B}}(a)(x \otimes y).$$

Proof. If $a, x, y \in A$ then

$$\begin{aligned} \delta_a[x, y] &= \Delta([x, y])(a \otimes 1) - \tau(\Delta([x, y])(a \otimes 1)) \\ &= \Delta(xy - yx)(a \otimes 1) - \tau(\Delta(xy - yx)(a \otimes 1)), \quad \text{and since } \Delta \text{ is a derivation,} \\ &= (x \otimes 1)\Delta(y)(a \otimes 1) + \Delta(x)(a \otimes y) - (y \otimes 1)\Delta(x)(a \otimes 1) - \Delta(y)(a \otimes x) \\ &\quad - \tau((x \otimes 1)\Delta(y)(a \otimes 1)) - \tau(\Delta(x)(a \otimes y)) + \tau((y \otimes 1)\Delta(x)(a \otimes 1)) + \tau(\Delta(y)(a \otimes x)). \end{aligned}$$

On the other side,

$$\begin{aligned} x \cdot \delta_a(y) &= (ad_x \otimes 1 + 1 \otimes ad_x)\delta_a(y) \\ &= (x \otimes 1)\delta_a(y) - \delta_a(y)(x \otimes 1) + (1 \otimes x)\delta_a(y) - \delta_a(y)(1 \otimes x) \\ &= (x \otimes 1)\Delta(y)(a \otimes 1) - \tau((1 \otimes x)\Delta(y)(a \otimes 1)) - \Delta(y)(ax \otimes 1) + \tau(\Delta(y)(a \otimes x)) \\ &\quad + (1 \otimes x)\Delta(y)(a \otimes 1) - \tau((x \otimes 1)\Delta(y)(a \otimes 1)) - \Delta(y)(a \otimes x) + \tau(\Delta(y)(ax \otimes 1)). \end{aligned}$$

We also have,

$$\begin{aligned} y \cdot \delta_a(x) &= (y \otimes 1)\Delta(x)(a \otimes 1) - \tau((1 \otimes y)\Delta(x)(a \otimes 1)) - \Delta(x)(ay \otimes 1) + \tau(\Delta(x)(a \otimes y)) \\ &\quad + (1 \otimes y)\Delta(x)(a \otimes 1) - \tau((y \otimes 1)\Delta(x)(a \otimes 1)) - \Delta(x)(a \otimes y) + \tau(\Delta(x)(ay \otimes 1)), \end{aligned}$$

then, by definition of bi-balanceator, we obtain (9). In the same way we obtain equation (10). \square

Theorem 3.2. *Let (A, μ, Δ) be an infinitesimal multiplier bialgebra. If the bi-balanceator of A is symmetric then the linear maps*

$$(11) \quad \delta : A \rightarrow \text{Der}(A, A \otimes A), \quad x \mapsto \delta_a(x) := (1 \otimes a)\Delta(x) - \tau((1 \otimes a)\Delta(x)),$$

$$(12) \quad \zeta : A \rightarrow \text{Der}(A, A \otimes A), \quad x \mapsto \zeta^a(x) := \Delta(x)(a \otimes 1) - \tau(\Delta(x)(a \otimes 1)),$$

are well defined, where $\delta_a := \delta(a)$ and $\zeta^a := \zeta(a)$. Moreover, if we define

$$T_1, T_2 : A^{\text{Lie}} \otimes A^{\text{Lie}} \rightarrow A^{\text{Lie}} \otimes A^{\text{Lie}} \text{ by } T_1(u \otimes v) = (1 \otimes u)\Delta(v) \text{ and } T_2(u \otimes v) = \Delta(u)(v \otimes 1),$$

then the collection $(A, m - m \circ \tau, \delta, \zeta, T_1, T_2)$ is a derivator Lie bialgebra.

Proof. Since the bi-balanceator of A is symmetric, by proposition 3.1, the maps δ and ζ are derivations that take values in the space of derivations of A^{Lie} with values in the A^{Lie} -module $A \otimes A$. Then, the only

that we have to prove ins that the generalized coJacobi condition is satisfied. In fact if $a, b, x \in A$ the using Sweedler notation

$$\begin{aligned}
 (13) \quad (\zeta^b \otimes Id)(T_1(a \otimes x)) &= (\Delta \otimes Id)((1 \otimes a)\Delta(x))(b \otimes 1 \otimes 1) \\
 &\quad - (\tau \otimes id)((\Delta \otimes Id)((1 \otimes a)\Delta(x))(b \otimes 1 \otimes 1)) \\
 &= x_{(1)(1)}b \otimes x_{(1)(2)} \otimes ax_{(2)} - x_{(1)(2)} \otimes x_{(1)(1)}b \otimes ax_{(2)}.
 \end{aligned}$$

Now,

$$\begin{aligned}
 (14) \quad (\delta_a \otimes Id)(\tau(T_2(x \otimes b))) &= (1 \otimes a \otimes 1)(\Delta \otimes Id)(\tau(\Delta(x)(b \otimes 1))) \\
 &\quad - (\tau \otimes id)((1 \otimes a \otimes 1)(\Delta \otimes Id)(\tau(\Delta(x)(b \otimes 1)))) \\
 &= x_{(2)(1)} \otimes ax_{(2)(2)} \otimes x_{(1)}b - ax_{(2)(2)} \otimes x_{(2)(1)} \otimes x_{(1)}b;
 \end{aligned}$$

then, since Δ is coassociative (1), then $x_{(1)(1)}b \otimes x_{(1)(2)} \otimes ax_{(2)} = x_{(1)}b \otimes x_{(1)(2)} \otimes ax_{(2)}$. It follows that

$$(Id + \sigma + \sigma^2)(\zeta^b \otimes Id)(T_1(a \otimes x)) = (Id + \sigma + \sigma^2)(\delta_a \otimes Id)(\tau(T_2(x \otimes b)))$$

□

Remark 3.1. The definition of the coproduct in equations (9) and (10) can be something annoying since the presence of the factors $(a \otimes 1)$ and $(1 \otimes a)$, but this is because the coassociativity constraints 1.1.

Definition 3.2. A *generalized infinitesimal multiplier bialgebra* (or generalized multiplier ε -bialgebra) is a quadruple $(A, *, \cdot, \Delta)$ where

- $(A, *)$ and (A, \cdot) are associative algebras;
- (A, \cdot) has non-degenerate product;
- $\mathbb{M}(A \otimes A, \cdot)$ is a $(A, *)$ -bimodule;
- $\Delta : A \rightarrow \mathbb{M}(A \otimes A, \cdot)$ is a coproduct on A , and
- $\Delta(a * b) = a \triangleright \Delta(b) + \Delta(a) \triangleleft b$. Where \triangleright and \triangleleft stand for the left and right actions of $(A, *)$ on the multipliers of $(A \otimes A, \cdot \otimes \cdot)$.

Remark 3.2. In definition 1.2 when $*$ coincide with \cdot we only write (A, \cdot, Δ) for the resultant structure.

Remark 3.3. It is clear that if (A, Δ) is an ε -bialgebra in the sense of Aguiar [A1], with nondegenerated product then, it is a multiplier infinitesimal bialgebra in the sense of definition (1.2). In fact, the only that we need to specify is the actions of A on the space of multipliers $\mathbb{M}(A \otimes A)$ and they are given by

$$a \triangleright \mu = (a \otimes 1)\mu \quad \text{and} \quad \mu \triangleleft a = \mu(1 \otimes a),$$

for every $a \in A$ and $\mu \in \mathbb{M}(A \otimes A)$.

Example 3.1. Infinite Cyclic Group

Let us consider $X = \{a\}$ and let $F = \langle X \rangle$ be the infinite cyclic group generated X (i. e. F is the infinite cyclic group). Let k be a field and let $A = kF$ be the group algebra of F over k . Define

$\Delta : kF \rightarrow kF \otimes kF$ by the following formulas,

$$\begin{aligned}\Delta(e) &= 0, \quad \Delta(a) = e \otimes e, \quad \Delta(a^{-1}) = -a^{-1} \otimes a^{-1}, \\ \Delta(a^n) &= (a \otimes 1)\Delta(a^{n-1}) + e \otimes a^{n-1}, \text{ for any } n > 1, \\ \Delta(a^{-n}) &= -(a^{-n} \otimes 1)\Delta(a^n)(1 \otimes a^{-n}), \text{ for any } n > 1.\end{aligned}$$

Lemma 3.3. *The collection (kF, m, Δ) is a ε -bialgebra.*

Proof. First, it is clear that (kF, Δ) is an associative coalgebra. In fact, an induction argument shows that the formulas given for Δ reduces to,

$$(15) \quad \Delta(a^{n+1}) = e \otimes a^n + a \otimes a^{n-1} + \cdots a^{n-1} \otimes a + a^n \otimes e, \quad \text{for all } n > 0,$$

$$(16) \quad \Delta(a^{-n}) = -(a^{-n} \otimes a^{-1} + a^{-(n-1)} \otimes a^{-2} + \cdots + a^{-2} \otimes a^{-(n-1)} + a^{-1} \otimes a^{-n}), \quad \text{for all } n > 0.$$

To prove the compatibility condition, between m and Δ we use induction again.

- Given $m > n > 0$ we have that,

$$\Delta(a^m a^{-n}) = \Delta(a^{m-n}) = (a \otimes e)\Delta(a^{m-n-1}) + e \otimes a^{m-n-1};$$

now, by induction hypothesis $\Delta(a^{m-n-1}) = (a^{m-1} \otimes e)\Delta(a^{-n}) + \Delta(a^m)(e \otimes a^{-n})$, then

$$\Delta(a^{m-n}) = (a^m \otimes e)\Delta(a^{-n}) + (a \otimes e)\Delta(a^{m-1})(e \otimes a^{-n}) + e \otimes a^{m-n-1}.$$

On the other hand,

$$\begin{aligned}& (a^m \otimes e)\Delta(a^{-n}) + \Delta(a^m)(e \otimes a^{-n}) \\ &= (a^m \otimes e)(-(a^{-n} \otimes e)\Delta(a^n)) + \Delta(a^m)(e \otimes a^{-n}) \\ &= -(a^{m-n} \otimes e)\Delta(a^n)(e \otimes a^{-n}) + (a \otimes e)\Delta(a^{m-1})(e \otimes a^{-n}) - (a^{m-1} \otimes e)\Delta(a^n)(e \otimes a^{-n}),\end{aligned}$$

and the conclusion follows.

- For $0 < n < m$ a similar argument shows that $\Delta(a^m a^{-n}) = (a^m \otimes e)\Delta(a^{-n}) + \Delta(a^m)(e \otimes a^{-n})$.

In conclusion, (kF, m, Δ) is an ε -bialgebra. \square

Let us denote by $K(F)$ the space of finitely supported complex valued functions defined on F . For any $n \in \mathbb{Z}$ we will write δ_n for the function defined over F that takes the value 1 at a^n and is equals to zero in other case. It is clear that every element $f \in K(F)$ can be written as the *finite* linear combination $\sum_{n \in \mathbb{Z}} f(a^n) \delta_n$. Over $K(F)$ we can define two structures of associative algebra, the first one is the classical structure defined by the formula $\delta_m \cdot \delta_n = \delta_{n,m} \delta_m$, and the second one is defined in the following

Lemma 3.4. *Let $*$: $K(F) \otimes K(F) \rightarrow K(F)$ be the bilinear map defined as follows*

- $\delta_m * \delta_n = \delta_{m+n+1}$, for every $m, n \in \mathbb{Z}_{\geq 0}$,
- $\delta_m * \delta_n = -\delta_{m+n+1}$, for every $m, n \in \mathbb{Z}_{< 0}$, and
- $\delta_m * \delta_n = 0$ otherwise.

With these map $K(F)$ becomes an associative nonunital algebra.

Proof. Straightforward. \square

Remark 3.4. It is clear that the space of multipliers of $(K(F) \otimes K(F), \cdot \otimes \cdot)$ is isomorphic to $C(F \times F)$, the space of all complex valued functions defined over F . Then $\mathbb{M}(K(F) \otimes K(F), \cdot \otimes \cdot)$ is a $(K(F), *)$ -module.

Proposition 3.5. *Let F be the free group in one generator and let us denote by $K(F)$ the space of finitely supported complex valued functions defined on F . With the same notation of the above paragraph we define*

$$(17) \quad \Delta : K(F) \rightarrow C(F \times F) \quad \text{by} \quad \Delta(f)(a^m, a^n) = f(a^{m+n}),$$

for every $f \in K(F)$ and $m, n \in \mathbb{Z}$. Then $(K(F), *, \cdot, \Delta)$ is a generalized multiplier ε -bialgebra.

Proof. The proof relies on the fact that $(K(F), *, \cdot, \Delta)$ is essentially dual to (kF, m, Δ) of lemma 3.3 and we don't include it here. \square

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